ISRAEL JOURNAL OF MATHEMATICS, Vol. 24, Nos. 3–4, 1976

A FUNCTIONAL CHARACTERIZATION OF CERTAIN MIXED VOLUMES

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ABSTRACT
A function over the convex cone $\mathcal{K}_n$ of convex bodies K in Euclidean n-space (where addition is vector addition, positive scalar multiplication is dilatation), which is linear over $\mathcal{K}_n$, increasing with respect to set inclusion, and zero at point bodies must be a mixed volume $V(K;\bar{K},p-1;\sigma_1,\cdots,\sigma_{n-p})$. Here $\bar{K}$, taken $p-1$ times, is in $\mathcal{K}_n$ and $\sigma_1,\cdots,\sigma_{n-p}$ are pairwise orthogonal unit segments spanning the orthogonal complement of the affine hull of $\bar{K}$.

Let $\mathcal{K}_n$ denote the class of all compact convex sets (convex bodies) in Euclidean n-dimensional space E^n. The topology in $\mathcal{K}_n$ is that induced by the Hausdorff metric. For non-negative λ_0,λ_1 and for K_0,K_1 in $\mathcal{K}_n$, the vector sum $\lambda_0K_0+\lambda_1K_1$ is the set of points $\lambda_0x_0+\lambda_1x_1$, where x_0 lies in K_0, x_1 lies in K_1; this vector sum is in $\mathcal{K}_n$. We write $\langle x,y\rangle$ for the inner product of x and y and $\|x\|=\sqrt{\langle x,x\rangle}$. B signifies the unit ball $\|x\|\leqq 1$. Finally, by the affine hull of a set we mean that translate of a subspace which contains the set and has least dimension.

This note characterizes certain functions over $\mathcal{K}_n$ as mixed volumes. To explain this, let $V(K)$ signify the volume of K in $\mathcal{K}_n$. Then $V(\lambda_1K_1+\cdots+\lambda_nK_n)$ for $\lambda_i\geqq 0$, K_i in $\mathcal{K}_n$ exists and is a homogeneous polynomial in $\lambda_1,\cdots,\lambda_n$ of degree at most n. We write the coefficient of the product $\lambda_1,\cdots,\lambda_n$ as $n!\,V(K_1,\cdots,K_n)$ with the understanding that $V(K_1,\cdots,K_n)$, called the mixed volume of these convex bodies, is symmetric in all its arguments. For details, see [3, pp. 38–41]. If $K_1,\cdots,K_{q_1}$ all equal K_1', $K_{q_1+1},\cdots,K_{q_2}$ all equal K_2' and so on, then we write $V(K_1,\cdots,K_n)$ as $V(K_1',q_1;\cdots;K_n',q_n)$. Here $q_1+\cdots+q_n=n$ and we suppress any q_j which equals one. Here is the main result.

† This work was supported in part by a grant from the U.S. National Science Foundation (NSF GP-28291).

Received November 1, 1975

THEOREM. *Assume that φ is a real-valued function over $\mathcal{K}_n$ which is not identically zero. Then φ satisfies:*

(i) *φ is linear: for non-negative λ_0, λ_1,*

$$\varphi(\lambda_0 K_0 + \lambda_1 K_1) = \lambda_0 \varphi(K_0) + \lambda_1 \varphi(K_1);$$

(ii) *φ is increasing: $K_0 \subseteq K_1$ entails $\varphi(K_0) \leqq \varphi(K_1)$;*

(iii) *$\varphi(t) = 0$ when t is a point;*

if and only if there is a $\bar{K}$ in $\mathcal{K}_n$, unique to within a translation, such that

$$\varphi(K) = V(K, \bar{K}; p-1; \sigma_1; \cdots; \sigma_{n-p}), \tag{1}$$

where $\sigma_1, \cdots, \sigma_{n-p}$ are segments of unit length, whose directions are mutually orthogonal and which span the orthogonal complement of the affine hull of $\bar{K}$.

PROOF. Suppose φ has the form (1). Then it is known that φ satisfies (i), (ii), (iii), see [3, pp. 40–41].

Before proving the sufficiency, we present some needed background. Let $\mathscr{C}$ signify the set of continuous functions on the unit sphere Ω in E^n; we give $\mathscr{C}$ the topology of uniform convergence on Ω. Extend each function F in $\mathscr{C}$ to $\bar{f}$ over E^n by the rule

$$\bar{f}(0) = 0, \qquad \bar{f}(x) = \|x\| f(x/\|x\|), \quad \text{for} \quad x \neq 0.$$

In $\mathscr{C}$ single out the subset $\mathscr{H}_n$ of those functions h whose extensions $\bar{h}$ are subadditive:

$$\bar{h}(x+y) \leqq \bar{h}(x) + \bar{h}(y).$$

The subset $\mathscr{H}_n$ is a cone in the linear space $\mathscr{C}$ and $\mathscr{H}_n + (-\mathscr{H}_n)$ is dense in $\mathscr{C}$, that is, for any $\varepsilon > 0$ and any f in $\mathscr{C}$ there are functions h_0, h_1 in $\mathscr{H}_n$ for which

$$\max_{u \in \Omega} |f(u) - (h_0(u) - h_1(u))| < \varepsilon.$$

For a proof see [4, p. 10].

Next, there is a one-to-one correspondence between $\mathscr{H}_n$ and $\mathcal{K}_n$ such that, if h, h_j in $\mathscr{H}_n$ correspond to K, K_j in $\mathcal{K}_n$, then $\{h_j\}$ converges to h if and only if $\{K_j\}$ converges to K. The correspondence is this: given K, h is the restriction to Ω of

$$\bar{h}(x) = \max_{y \in K} \langle x, y \rangle;$$

given h, K is the intersection of all those half-spaces of points x for which

$$\langle x, u\rangle \leqq h(u), \quad u \in \Omega.$$

In short $\bar{h}$ is the support function of K, see [3, pp. 26–28]. Note that $\lambda_0 h_0 + \lambda_1 h_1$, for $\lambda_j \geqq 0$, corresponds to $\lambda_0 K_0 + \lambda_1 K_1$ when h_j corresponds to K_j, $j = 0, 1$.

We define the functional $\bar{\varphi}$ over $\mathscr{H}_n$ by

$$\bar{\varphi}(h) = \varphi(K). \tag{2}$$

Now for the sufficiency proof. In (ii) choose K_0 to be a point t in K_1 and apply (iii). This shows φ is non-negative. It is also translation invariant. To see this, set

$$\lambda_0 = \lambda_1 = 1, \quad K_1 = t$$

in (i). In view of (iii) this gives

$$\varphi(K_0 + t) = \varphi(K_0) \tag{3}$$

as claimed.

Next we prove φ is continuous. Suppose $\{K_j\}$ is an infinite sequence from K_n which converges to K. First we take the case in which K is not degenerate, that is, has interior points, and so contains a ball $2\varrho B$, $\varrho > 0$, possibly after a suitable translation which cannot affect the values of φ. For all but a finite set of index-values j we must have $K_j \supseteq \varrho B$. From here on we omit those K_j for which this inclusion fails. The convergence of $\{K_j\}$ to K implies there is a sequence $\{\varepsilon_j\}$ of positive numbers, convergent to zero, such that

$$K \subseteq K_j + \varepsilon_j B \subseteq (1 + \varepsilon_j/\varrho)K_j, \quad K_j \subseteq K + \varepsilon_j B \subseteq (1 + \varepsilon_j/\varrho)K.$$

We apply (i), (ii) to obtain

$$\varphi(K) \leqq (1 + \varepsilon_j/\varrho)\varphi(K_j) \leqq (1 + \varepsilon_j/\varrho)^2\varphi(K).$$

It follows that $\{\varphi(K_j)\}$ converges to $\varphi(K)$.

If K has no interior points, consider $K + \lambda B$, $\lambda > 0$, which does. The sequence $\{K_j\}$ converges to K if and only if $\{K_j + \lambda B\}$ converges to $K + \lambda B$. By our continuity result in the non-degenerate case and by (i), we have for each $\lambda > 0$:

$$\varphi(K + \lambda B) = \varphi(K) + \lambda\varphi(B), \quad \varphi(K_j + \lambda B) = \varphi(K_j) + \lambda\varphi(B);$$

$\{\varphi(K_j) + \lambda\varphi(B)\}$ converges to $\varphi(K) + \lambda\varphi(B)$. Hence $\{\varphi(K_j)\}$ converges to $\varphi(K)$ and the proof of the continuity of φ is complete.

Observe the implications of our results about φ for the functional $\bar{\varphi}$ defined in (2): over $\mathscr{H}_n$, $\bar{\varphi}$ is continuous and, for non-negative λ_0, λ_1

$$\bar{\varphi}(\lambda_0 h_0 + \lambda_1 h_1) = \lambda_0\bar{\varphi}(h_0) + \lambda_1\bar{\varphi}(h_1).$$

We extend $\bar{\varphi}$ to $\mathscr{C}$ by the rules

$$\bar{\varphi}(h-h')=\bar{\varphi}(h)-\bar{\varphi}(h'),$$

$$\bar{\varphi}(\lim_{j\to\infty}(h_j-h'_j))=\lim_{j\to\infty}\bar{\varphi}(h_j-h'_j),$$

for h, h', h_j, h'_j all in $\mathscr{H}_n$ and assuming $\{h_j-h'_j\}$ converges. It is easy to show that the extended functional $\bar{\varphi}$ is linear. Details of this sort of argument appear in [1, pp. 959–961]. Also, for non-negative f in $\mathscr{C}$

$$\bar{\varphi}(f)\geqq 0.$$

To see this last, we use (ii) and the fact that $K\subseteq K'$ for K, K' in $\mathscr{K}_n$ if and only if $h(u)\leqq h'(u)$ over Ω for the corresponding h, h' in $\mathscr{H}_n$.

As a positive linear functional over $\mathscr{C}$, $\bar{\varphi}$ has the representation

$$\bar{\varphi}(f)=\int_\Omega f(u)\mu(d\omega(u))/n,$$

for some unique, non-negative measure μ, defined over the Borel sets $\mathscr{B}$ of Ω. This is a consequence of the representation theorem of F. Riesz, see [6, pp. 243–248]. In particular, if h in $\mathscr{H}_n$ is the support function of K in $\mathscr{K}_n$ then from (2)

$$\varphi(K)=\int_\Omega h(u)\mu(d\omega(u))/n.$$

The support function of the translate $K+t$ is $h(u)+\langle t,u\rangle$. This, together with the translation invariance of φ shows that μ satisfies

$$\int_\Omega \langle t,u\rangle\mu(d\omega(u))/n=0 \tag{4}$$

for all t in E^n. Let L be the linear subspace of least dimension for which $L\cap\Omega=\Omega'$ contains the support of μ. Write p for the dimension of L. By the hypothesis of the theorem $p>0$.

Any $(p-1)$-dimensional subspace L' in L determines two open hemispheres Ω_1,Ω_2 on Ω', separated by $L'\cap\Omega$. We claim $\mu(\Omega_1)$, $\mu(\Omega_2)$ are both positive. One of them certainly is since the support of μ does not lie in $L'\cap\Omega$. If $\mu(\Omega_1)$ were zero, we could choose t so that

$$\langle t,u\rangle>0 \quad \text{for} \quad u\in\Omega_2,$$

and then we would have

$$\int_{\Omega} \langle t, u\rangle \mu(d\omega(u)) > 0$$

which contradicts (4). We may repeat the foregoing with Ω_1, Ω_2 interchanged. This shows μ is positive on every open hemisphere of Ω'.

A digression is necessary. For the moment we work entirely in L, viewed as a Euclidean space E^p; Ω' is the unit sphere centred at the origin, and we write $\mathcal{B}'$ for the Borel sets of Ω'. By the area function of a convex body $\bar{K}$ in L we mean that unique measure $s(\bar{K}; \omega')$, defined for all ω' in $\mathcal{B}'$ as follows. Let σ be the set of points on the boundary of $\bar{K}$ which lie in support hyperplanes to $\bar{K}$, whose outer normals fall in ω'. The $(p-1)$-dimensional measure of σ is $s(\bar{K}; \omega')$. If $\bar{K}$ is not degenerate, then $s(\bar{K}; \omega')$ is positive on the open hemispheres of Ω' and

$$\int_{\Omega'} \langle t, u\rangle s(\bar{K}; d\omega'(u)) = 0$$

for all t in L. Conversely, any measure satisfying these two conditions is the area function of a convex body $\bar{K}$, unique up to a translation, see [4, pp. 16–17], [2, pp. 35–39].

This, with what we have proved about μ shows that there is a convex body $\bar{K}$ in L such that

$$s(\bar{K}; \omega') = \mu(\omega') \tag{5}$$

for ω' in $\mathcal{B}'$; $\bar{K}$ is unique up to a translation in L.

From here on we go back to E^n; we write the area function of a convex body K as $S(K; \omega)$, where ω is in $\mathcal{B}$. Just as mixed volumes arise as coefficients in the polynomial $V(\lambda_1 K_1 + \cdots + \lambda_n K_n)$, so do we obtain mixed area functions $S(K_1, \cdots, K_{n-1}; \omega)$ as coefficients in the polynomial $S(\lambda_1 K_1 + \cdots + \lambda_{n-1} K_{n-1}; \omega)$ of degree $n-1$; of course these coefficients are measures over $\mathcal{B}$. More precisely, for $\lambda_j \geqq 0$, K_j in $\mathcal{K}_n$, $j = 1, 2, \cdots, n-1$, $(n-1)! S(K_1, \cdots, K_{n-1}; \omega)$ is the coefficient of the product $\lambda_1 \cdots \lambda_n$ in $S(\lambda_1 K_1 + \cdots + \lambda_{n-1} K_{n-1}; \omega)$ with the understanding that $S(K_1, \cdots, K_{n-1}; \omega)$ is symmetric in $K_1, \cdots, K_{n-1}$. As we did for mixed volumes, we write $S(K'_1, q_1; \cdots; K'_{n-1}, q_{n-1}; \omega)$ if $K_1, \cdots, K_{q_1}$ all equal K'_1, $K_{q_1+1}, \cdots, K_{q_2}$ all equal K'_2 and so on. Here $q_1 + \cdots + q_{n-1} = n-1$ and we suppress any q_j which equals one. We remark that the mixed volume $V(K, K_1, \cdots, K_{n-1})$ has the representation

$$V(K, K_1, \cdots, K_{n-1}) = \int_{\Omega} h(u) S(K_1, \cdots, K_{n-1}; d\omega(u))/n. \tag{6}$$

Two more facts: $S(K_1, \cdots, K_{n-1}; \omega)$ satisfies (4) when substituted for μ; in $S(K_1, \cdots, K_{n-1}; \omega)$ we may replace any K_j by any one of its translates without altering the measure. For details see [4, pp. 21–25], [1, pp. 959–967].

This prepares us for the final part of the proof of the theorem. Let $\sigma_1, \cdots, \sigma_{n-p}$ be segments of unit length whose directions are orthogonal in pairs and which span the orthogonal complement L_1 of L in the sense that $\sigma_1 + \cdots + \sigma_{n-p}$ has relative interior points in some translate of L_1. We claim

$$S(\bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}; \omega) = \mu(\omega) \tag{7}$$

over $\mathscr{B}$. When (7) has been proved, in view of (6) and (1), the theorem follows.

To prove (7) it suffices to show, first, that $S(\bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}; \omega)$ has its support in Ω' as does μ, and second, that (7) holds when ω is in $\mathscr{B}'$.

The mixed volume $V(K_1, \cdots, K_n)$ is positive precisely when there are segments $\sigma'_1, \cdots, \sigma'_n$, in $K_1, \cdots, K_n$ respectively, which span E^n. By (6) it follows that

$$V(K; \bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}) > 0$$

if and only if K contains a segment in L. Put another way,

$$V(K; \bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}) = 0 \tag{8}$$

if and only if K is in the span L_1 of $\sigma_1, \cdots, \sigma_{n-p}$. In computing the left side of (8) by means of (6), we may assume that the support function $h(u)$ of K is non-negative over Ω by translating K so as to contain the origin 0. But K is in L_1 if and only if $h(u)$ vanishes for u orthogonal to L_1. In short, (8) holds precisely when $h(u)$ vanishes over L. Thus the support of $S(\bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}; \omega)$ is in Ω' as claimed.

Consider the support hyperplane $\Pi(u')$ to the convex body

$$K = \lambda\bar{K} + \lambda_1\sigma_1 + \cdots + \lambda_{n-p}\sigma_{n-p}, \ \lambda \geqq 0, \lambda_j \geqq 0,$$

with outer unit normal u' in ω' of $\mathscr{B}'$. In the statements which follow we neglect translations; area and mixed area functions are translation-invariant. Since each segment σ_j is perpendicular to u', $\Pi(u') \cap \sigma_j$ is σ_j. Consequently $\Pi(u')$ meets K in the set

$$(\lambda\bar{K} \cap \Pi(u')) + \lambda_1\sigma_1 + \cdots + \lambda_{n-p}\sigma_{n-p}.$$

Next, since each σ_j has length one, is orthogonal to every other σ_i and is orthogonal to L, we find

$$S(K; \omega') = (n-1)!\lambda^{p-1}\lambda_1 \cdots \lambda_{n-p} s(\bar{K}; \omega').$$

We have used here the fact that the union over u' in ω' of the sets $\bar{K} \cap \Pi(u')$, $u' \in \omega'$, has area, as a set in L, equal to $s(\bar{K}; \omega')$. Because $S(K; \omega)$ is a monomial, it must equal

$$(n-1)!\lambda^{p-1}\lambda_1 \cdots \lambda_p S(\bar{K}, p-1; \sigma_i; \cdots; \sigma_{n-p}; \omega').$$

This, with our last equation and with (5) finishes the proof of (7) and of the theorem.

The following remarks amplify the theorem.

Clearly if in place of (ii) we assume only that φ is monotone under set inclusion, then this will include the possibility that it is $-\varphi$ which has the representation claimed in the theorem. Next, (i) and (iii) imply the translation-invariance (3) of φ. However, (i) and (3) imply (iii) and so we may substitute (3) for (iii).

It is possible to find the subspace L and the convex body $\bar{K}$, used in the representation (1), more directly from φ. The conditions on K for (8) to hold show how to find L: take the orthogonal complement of the largest subspace L_1, such that $\varphi(K) = 0$ whenever K is in L_1. The support of $S(\bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}; \omega')$ is in Ω'. Also the restriction of $h(u)$ to L is the support function of the image K' of K under orthogonal projection onto L. Hence for any K in $\mathcal{K}_n$

$$\varphi(K) = \int_{\Omega'} h(u')S(\bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p}; d\omega(u'))/n = \varphi(K').$$

Thus the range of $\varphi(K)$ is determined by just those K in L.

In this paragraph once again we work entirely in L, viewed as a Euclidean space E^p. We write $v(K)$, $v(K; \bar{K}, p-1)$ for the volume of K and the indicated mixed volume. In this notation

$$\varphi(K) = v(K; \bar{K}, p-1). \tag{9}$$

Minkowski's inequality, [3, p. 91] reads

$$v^p(K; \bar{K}, p-1) \geqq v(K)v^{p-1}(\bar{K}),$$

with equality if and only if K and $\bar{K}$ are homothetic. This, with (9), shows that

$$\min_{v(K)=1} \varphi(K) = \mu$$

exists. Further, if we write K^* for the unique convex body at which this minimum is attained, then $\bar{K} = \mu^{1/(p-1)}K^*$.

The final comment concerns a special case of the theorem. Suppose that, in addition to (i), (ii), (iii), φ is rotation invariant, that is, if $\mathcal{R}K$ is the image of K under the rotation $\mathcal{R}$ of E^n, then

$$\varphi(\mathcal{R}K) = \varphi(K). \tag{10}$$

Since (3) can replace (iii), our assumptions on φ are now (i), (ii) and the following: φ is rigid motion invariant. In this case, there is a constant $c \geqq 0$ such that $\varphi(K)$ is c times the mean width $W_{n-1}(K)$. This result is due to Hadwiger [5, p. 213]; note that the meaning of linearity in (i) differs slightly from that of Hadwiger [5].

In

$$\varphi(\mathcal{R}K) = V(\mathcal{R}K; \bar{K}, p-1; \sigma_1; \cdots; \sigma_{n-p})$$

replace $K, \bar{K}, \sigma_1, \cdots, \sigma_{n-p}$ by $\mathcal{R}^{-1}K, \mathcal{R}^{-1}\bar{K}, \mathcal{R}^{-1}\sigma_1, \cdots, \mathcal{R}^{-1}\sigma_{n-p}$. This does not change the value of the mixed volume [3, p. 40] and, with (10), gives

$$\varphi(K) = V(K; \mathcal{R}^{-1}\bar{K}, p-1; \mathcal{R}^{-1}\sigma_1; \cdots; \mathcal{R}^{-1}\sigma_{n-p})$$

over $\mathcal{K}_n$ and for all $\mathcal{R}$. By our theorem, $\bar{K}$ is unique to within a translation and so $\bar{K} = \mathcal{R}\bar{K}$ for all $\mathcal{R}$. Hence $p = n$ and $\bar{K}$ is a ball ϱB; the special mixed volume $V(K; B, n-1)$ is $W_{n-1}(B)$. This proves Hadwiger's theorem with $c = 2\varrho^{n-1}/V(B)$.

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